A Subjective Approach to Quantum Probability: the current state of affairs and future directions.

Based on a paper with Eran Shmaya

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An overview

- An interpretation of quantum probability as an analogue to de Finetti/Savage theory of (classical) subjective probability;
- Subjective approaches interpret the probability of a proposition as the 'degree of belief' in the validity of the proposition;
- Derive quantum probabilities from a weak order over propositions about outcome of measurements in a physical system;
- These are represented as subspaces of a finite-dimensional Hilbert space;
- The weak order represents the preference of a rational agent over bets on the outcome of the measurements;
- Rationality - captured by several axioms.

A classical system

- *Phase space* – with N particles it is a subset $\mathcal{S}$ of $\mathbb{R}^{\ell N}$;
- Physical property – a subset of $\mathcal{S}$;
- Logical operations over physical properties – negation, disjunction and conjunction – represented by the corresponding set-theoretic operations (i.e., complement, union and intersection) over subsets of $\mathcal{S}$;
- Example: 'The energy is between 10J and 12J – represented by a subset X of $\mathcal{S}$
  'The energy is between 12J and 15J – represented by a subset Y
  'The energy is between 10J and 15J – represented by the union $X \cup Y$.

A classical system – logical operators

- X and Y – two properties;
- X and Y are *mutually exclusive* if both cannot be true at the same instant – X and Y are disjoint;
- X *implies* Y if X is a subset of Y.

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- X and Y are *mutually exclusive* if both cannot be true at the same instant – X and Y are disjoint;
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- $\mathcal{L}_c(\mathcal{S})$ – the *lattice of physical properties*. Consists of the set of subsets of $\mathcal{S}$, equipped with the logical operations.

A quantum system

- *Phase space* – separable Hilbert space $\mathcal{H}$;
- Physical property – a closed subspace of $\mathcal{H}$;
- $\mathcal{L}_q(\mathcal{H})$ – the *lattice of physical properties* that corresponds to $\mathcal{H}$;
- $\mathcal{L}_q(\mathcal{H})$ is the set of closed subspaces of $\mathcal{H}$;
- Logical operators:
  (i) Negation of a property – its orthogonal complement
  (ii) Mutually exclusive properties – orthogonal subspaces
  (iii) X implies Y – represented by $X \subseteq Y$.

A difference

- In the classical – $X \cup Y$ is always defined;
- In other words, any two properties are compatible properties;
- Not in the quantum case;
- Two properties $X, Y \in \mathcal{L}_q(\mathcal{H})$ – compatible iff $\Pi_X \Pi_Y = \Pi_X \Pi_Y$ (Π_X and Π_Y are the orthogonal projection on X and Y);
- E.g., mutually exclusive properties are compatible;
- Disjunction of compatible properties – the algebraic span conjunction – intersection;
- $\mathcal{L}_q(\mathcal{H})$ is an *ortho-modular lattice*.

Summary of analogies

Logic	Classical Rep.	Quantum Rep.
Propositions	Subsets of $\mathcal{S}$	Subspaces of $\mathcal{H}$
FALSE (always)	ϕ	$\{0\}$
TRUE (tautology)	Ω	$\mathcal{H}$
negation	X^c	$X^\perp$
logical implication	$X \subseteq Y$	$X \subseteq Y$
mutual exclusiveness	$X \cap Y = \phi$	$X \perp Y$
disjunction	$X \cup Y$	$X \bigoplus Y$

$\mathcal{H}$ is a finite dimensional Hilbert space

Information: probability and belief

- In the classical system – probability distribution over $\mathcal{S}$, i.e. a real-valued function $\mathbb{P}$ over $\mathcal{L}_q(\mathcal{S})$;
- That is,
  1. $\mathbb{P}(\mathcal{S}) = 1$
  2. $\mathbb{P}(X) \geq 0$ for every $X \subseteq \mathcal{S}$
  3. $\mathbb{P}(X \cup Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ if $X \cap Y = \varnothing$.
- In the quantum system – defined similarly, replacing the classical representation of property with the quantum one;
- A *quantum probability over $\mathcal{H}$* – a real valued function P over subspaces of $\mathcal{H}$ such that
  1. $\mathbb{P}(\mathcal{H}) = 1$
  2. $\mathbb{P}(X) \geq 0$ for every subspace X of $\mathcal{H}$
  3. $\mathbb{P}(X \bigoplus Y) = \mathbb{P}(X) + \mathbb{P}(Y)$ for every orthogonal subspaces X, Y.

Gleason Theorem

Gleason's Theorem characterizes all the quantum probabilities over a separable Hilbert space.

Theorem

Let $\mathcal{H}$ be a separable Hilbert space such that $\dim(\mathcal{H}) \geq 3$ *and* $\mathbb{P}$ *a quantum probability over* $\mathcal{H}$*. Then, there exists a trace-1 positive semi-definite Hermitian operator* ρ *over* $\mathcal{H}$ *such that* $\mathbb{P}(X) = tr(\rho \Pi_X)$ *for every subspace* X*.*

One important consequence of Gleason's Theorem is that there exists no quantum probability $\mathbb{P}$ over $\mathcal{H}$ such that $\mathbb{P}(X) = 0$ or $\mathbb{P}(X) = 1$ for every subspace X of $\mathcal{H}$. That is, the observer can never be certain (assign probability 0 or 1) about the validity of every property of the system. This is in sharp contrast with the classical case.

Quantum physics terminology

- Positive semi-definite Hermitian operator with trace 1 – *density operator*;
- That is, information about a physical system is given by a density operator;
- *Measurement* – linear function from the set of probability distributions to the simplex $\Delta(O)$ where O is a finite set of outcomes.

Subjective probability

- Savage derives probabilities from a weak order over properties;
- Weak order obtained - comparing the amount of money that the observer is willing to invest in a gamble that offers $1000 should it turn out (after performing a suitable measurement) true;
- Weak order satisfies several assumptions – representing rationality.

Savage's Theorem – classical system

- $\mathcal{S}$ – state space;
- $\preceq$ – weak order over $\mathcal{L}_c(\mathcal{S})$;
- Provides sufficient conditions for the existence classical *representation* $\mathbb{P}$ over $\mathcal{S}$: $X \preceq Y$ iff $\mathbb{P}(X) \leq \mathbb{P}(Y)$ for every $X, Y \in \mathcal{L}_c(\mathcal{S})$;
- *de-Finetti's Axiom* – $X, Y, Z \in \mathcal{L}_c(\mathcal{S})$ such that $X \cap Z = Y \cap Z = \phi$ then $X \preceq Y \Leftrightarrow X \cup Z \preceq Y \cup Z$;
- In logical terms – if x, y, z are properties such that the pairs x, z and y, z are mutually exclusive then $x \preceq y \Leftrightarrow x \vee z \preceq y \vee z$.

Savage's Theorem – quantum system

- We use this same axiom to derive a representation of a weak order over $\mathcal{L}_q(\mathcal{H})$ by a quantum probability distribution over $\mathcal{H}$;
- Note that in de-Finetti axiom the disjunction of properties is only used for mutually exclusive properties, which are always compatible.

The cancelation condition – classical framework

Proposition

Let $\preceq$ be a weak order over subsets of a finite set Ω. For $A \subseteq \Omega$, denote by $\mathbf{1}_A$ the indicator function of A. Then there exists an additive probability measure μ over Ω such that $A \preceq B \Leftrightarrow \mu(A) \leq \mu(B)$ for every $A, B \subseteq \Omega$ if and only if the following conditions hold:

1. *For every $A \subseteq \Omega$, $\varnothing \preceq A$.*
2. *$\varnothing \prec \Omega$.*
3. *For every n, if $A_1, \ldots, A_n, B_1, \ldots, B_n$ are subsets of Ω such that $\sum_{i=1}^{n} \mathbf{1}_{A_i} = \sum_{i=1}^{n} \mathbf{1}_{B_i}$ and $A_i \preceq B_i$ for every i, then $A_i \sim B_i$ for every i.*

The cancelation condition – quantum framework

The likelihood order $\preceq$ over $\mathcal{L}_q(\mathcal{H})$.

Definition

The likelihood order $\preceq$ satisfies the *cancelation condition* if: for every $2n$ subspaces of $\mathcal{H}$, $A_1, \ldots, A_n, B_1, \ldots, B_n$, and n numbers $\alpha_1, ..., \alpha_n > 0$,

$$\sum_{i=1}^{n} \alpha_i \Pi_{A_i} = \sum_{i=1}^{n} \alpha_i \Pi_{B_i} \text{ and } A_i \preceq B_i,\ \ i = 1, ..., n$$

imply

$$A_i \sim B_i \text{ for every } i = 1, ..., n.$$

Continuity

The cancelation condition by itself is not sufficient to ensure the existence of a representative measure.

Definition

The likelihood order $\preceq$ is *lower semi-continuous* if, for every subspace B, the set of the subspaces A such that $A \prec B$ is open (with respect to Hausdorff topology of subsets of $\mathcal{H}$).

Theorem

Let $\preceq$ be a likelihood order. There exists a quantum probability measure that represents $\preceq$ if and only if the following conditions are satisfied:

1. $\{0\} \preceq A$ For every subspace A of $\mathcal{H}$;
2. $\{0\} \prec \mathcal{H}$;
3. $\preceq$ is lower semi-continuous;
4. $\preceq$ satisfies the cancelation condition.

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- Cancelation: let $A_1, \ldots, A_n, B_1, \ldots, B_n$ (subspaces) with

$$\sum_{i=1}^{n} \alpha_i \Pi_{A_i} = \sum_{i=1}^{n} \alpha_i \Pi_{B_i}$$

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- Thus, $\mu(A_i) = \mu(B_i)$ implying $A_i \sim B_i$.

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- For $A \prec B$, lower semi-continuity $\Rightarrow \mathrm{tr}(\Pi_A T) > \mathrm{tr}(\Pi_B T)$);
- $\forall A, \mathrm{tr}(\Pi_A T) \geq 0$ since $\{0\} \preceq A$. Therefore, T is positive semidefinite;
- Finally, $\{0\} \prec \mathcal{H} \Rightarrow \mathrm{tr}(T) > 0$. Normalize: $T' = \frac{T}{\mathrm{tr}(T)}$. T' is positive semidefinite and $\mathrm{tr}(T') = 1$ and represents $\preceq$.

The cancelation condition – a flaw

The cancelation condition (even in the classical framework) is difficult to justify. It is desirable to derive a probability representation of a likelihood order over linear subspaces from more plausible assumptions.

de-Finetti's and Other Axioms

- **de-Finetti's Axiom**: For every linear subspaces A, B, C of $\mathcal{H}$, if $A \perp C$ and $B \perp C$, then $A \preceq B$ iff $A + C \preceq B + C$. ;
- In the classical framework it easily follows from **de-Finetti's axiom** that if $A \preceq B$ then $B^c \preceq A^c$. In the quantum framework, we need to require it explicitly;
- **Negation**: For every two linear subspaces A, B of $\mathcal{H}$, if $A \preceq B$ then $B^\perp \preceq A^\perp$;
- **Monotonicity**: For every subspace A of $\mathcal{H}$, $\{0\} \preceq A$;
- **Separability**: There is a countable set of subspaces, $\mathcal{A}$, such that for any two subspaces B and C such that $B \prec C$, there is $A \in \mathcal{A}$ that satisfies $B \preceq A \preceq C$;
- Debreu: **Separability** is necessary for $\preceq$ in order to be represented by a real function (not necessarily a measure).

Example: **de-Finetti's axiom** and not **Negation**.

- $\mathcal{H}$ be $\mathbb{R}^3$.
- Let p be the northern pole of the unit ball, E be the equator, and define $\mu(u) = \langle p, u \rangle^2$ for any unit vector u;
- Let $\succeq'$ be a monotonic weak order on E;
- Define $\preceq$ as follows: If A and B are two subspaces of different dimensions. Then $A \succ B$ if $\dim(A) > \dim(B)$;
- If $A = \text{span}(u)$ and $B = \text{span}(v)$, where u and v are unit vectors, then $A \succeq B$ either when $u \notin E$ and $\mu(u) \geq \mu(v)$ or when $u, v \in E$ and $u \succeq' v$;
- Finally, if A and B are two-dimensional subspaces, then $A \succeq B$ either when $p \notin B$ and $\|\Pi_A(p)\|^2 > \|\Pi_B(p)\|^2$ or when $p \in A \cap B$ and $A \cap E \succeq' B \cap E$;
- The weak order $\preceq$ preserves **de-Finetti's axiom** but since $\succeq'$ does not preserve **Negation** on E, so $\succeq$ does not on $\mathcal{H}$.

Pure States

- *Pure states* – $\mu(A) = \|\Pi_A(p)\|^2$ for some unit vector p;
- From the physical point of view – he most important probabilities;
- From Gleason's Theorem: pure states are the extreme points of the convex set of quantum probabilities;
- From a decision theoretic point of view, pure states correspond to situations of, where there exists a maximal measurement (i.e., a complete set of orthogonal one-dimensional subspace) whose outcome the agent can predict with certainty;
- It is clear that if μ is a pure state representing $\preceq$, then the one-dimensional subspace spanned by p is equivalent to $\mathcal{H}$.
- We say that $\preceq$ is *non-trivial* if there exists a subspace A such that $\{0\} \prec A$.

Pure states – the result

Theorem

Let $\preceq$ be a weak order over subspaces of a finite dimensional real-Hilbert space that satisfies **de-Finetti's axiom**, **Negation**, **Monotonicity** *and* **Separability**. *Assume that there exists a one-dimensional subspace P such that $P \sim \mathcal{H}$. Let p be a unit vector in P. If $\preceq$ is non-trivial, then $\preceq$ is represented by the pure state $\mu(A) = \|\Pi_A(p)\|^2$.*

Note: Continuity is not assumed in the theorem

The uniform measure

The only quantum probability measures over a finite dimensional Hilbert space $\mathcal{H}$ which receives discrete values is given by the *uniform measure*, $\mu(A) = \frac{\dim(A)}{\dim(\mathcal{H})}$. It turns out that this is the case characterized by the property that all one-dimensional subspaces are equally likely.

proposition

Let $\preceq$ be a weak order over subspaces of a finite dimensional Hilbert space that satisfies **de-Finetti's axiom**. If all one-dimensional subspaces are equivalent, then either $\preceq$ is trivial (i.e. $\{0\} \sim A$ for every subspace A of $\mathcal{H}$) or $\preceq$ is represented by the uniform measure.

The uniform measure – the main result

Definition

The likelihood order $\preceq$ is *continuous over one-dimensional subspaces* if for every unit vector v the sets $\{u;\ u \prec v\}$ and $\{u;\ v \prec u\}$ are open.

The result stated in $\mathbb{R}^3$ (easily to extend to any finite-dimensional Hilbert space).

Theorem

Let $\preceq$ be a weak order over $\mathbb{R}^3$ that satisfies the standard assumptions. Assume that $\preceq$ is continuous over one-dimensional subspaces. If u_1, u_2, u_3 is a basis (not necessarily orthogonal) that satisfies $u_1 \sim u_2 \sim u_3 \sim m$, where m is a minimum of $\preceq$, then $x \sim m$ for every unit vector x.

Open problem and future directions

- We have representations only in the extreme cases: one state and uniform. Can you have a presentation of the general case without cancellation?
- Borrow concepts and ideas from classical game theory and decision theory to the quantum realm. E.g., Blackwell's comparison of experiments (Shmaya).

Summary

- A qualitative approach;
- The primitive - a likelihood order over properties (subspaces);
- The goal is to represent it by a (quantum) measure: subjective probability;
- With cancellation – characterization exits;
- Without cancellation – only in two cases.